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# $d$-Variate Boolean Interpolation

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## INTRODUCTION

It has been shown in several papers [2, 3, 7, 8] that Boolean methods are useful for the construction of bivariate (and trivariate) interpolation schemes which are of importance in the finite element method. Our objective in this paper is to treat (for arbitrary $d \in \mathbb{N}$) a class of $d$-variate polynomial Lagrange interpolation schemes (including interpolation on $d$-simplices) which can be constructed by means of Boolean methods. In particular we present a simple representation formula for the projector of $d$-variate Boolean interpolation. Based on this formula explicit expressions for the fundamental functions of $d$-variate Boolean interpolation are derived.

## 1. $d$-VARIATE TENSOR PRODUCT INTERPOLATION

We begin by considering a rectangular domain

$$D = ]a_1', a_1''[ \times \cdots \times ]a_d', a_d''[ \subset \mathbb{R}^d;$$

we denote by $C(\bar{D})$ the algebra of real valued continuous functions of $d$ independent variables $x_1,..., x_d$ defined on $\bar{D}$.

Next we consider $d$ injective real sequences

$$(x_{i,u})_{i \in \mathbb{N}} \qquad (u = 1,..., d)$$

satisfying

$$(x_{i_1,1},..., x_{i_d,d}) \in \bar{D}$$
$$(i_u \in \mathbb{N};\ u = 1,..., d).$$



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Furthermore we assume that there are $d$ strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$:

$$a_u(m) < a_u(m+1), \qquad a_u(m) \in \mathbb{N}$$
$$(m \in \mathbb{N},\ u = 1, \dots, d).$$

Next we define the functions $f^m_{i,u}$:

$$f^m_{i,u}(x_1, \dots, x_d) = \prod_{\substack{k=1 \\ k \neq i}}^{a_u(m)} (x_u - x_{k,u})/(x_{i,u} - x_{k,u})$$
$$(i = 1, \dots, a_u(m);\ m \in \mathbb{N};\ u = 1, \dots, d).$$

The functions $f^m_{i,u}$ are $d$-variate extensions of univariate fundamental Lagrange polynomials. Note that

$$f^m_{i,u}(x_1, \dots, x_{j,u}, \dots, x_d) = \delta_{i,j}$$
$$(i, j = 1, \dots, a_u(m);\ u = 1, \dots, d;\ m \in \mathbb{N}). \tag{1.1}$$

Now we define $d$-variate extensions of univariate Lagrange interpolation operators:

$$P^m_u(f)(x_1, \dots, x_d) = \sum_{i=1}^{a_u(m)} f(x_1, \dots, x_{i,u}, \dots, x_d)\, f^m_{i,u}(x_1, \dots, x_d)$$
$$(m \in \mathbb{N};\ u = 1, \dots, d). \tag{1.2}$$

It is easily seen that the linear operator

$$P^m_u \in L(C(\bar{D}))$$

is also a projector:

$$P^m_u P^m_u = P^m_u \qquad (m \in \mathbb{N};\ u = 1, \dots, d). \tag{1.3}$$

Also the projectors $P^m_u$, $P^n_v$ commute:

$$P^m_u P^n_v = P^n_v P^m_u \qquad (m, n \in \mathbb{N};\ u, v = 1, \dots, d). \tag{1.4}$$

Therefore

$$\mathbb{P} = \{P^m_u \colon m \in \mathbb{N},\ u = 1, \dots, d\} \tag{1.5}$$

is a set of commuting projectors on $C(\bar{D})$.

It has been shown [4, 5] that there is a (maximal) Boolean algebra $\mathbb{P}''$ of projectors in $L(C(\bar{D}))$ containing the set $\mathbb{P}$. We briefly recall the construction of $\mathbb{P}''$.

First we introduce the set

$$\mathbb{P}' = \{Q \in L(C(\bar{D})) \colon Q^2 = Q,\ QP = PQ\ (P \in \mathbb{P})\}. \tag{1.6}$$

Obviously, we have

$$\mathbb{P} \subset \mathbb{P}',$$
$$0, I \in \mathbb{P}',$$
$$Q \in \mathbb{P}' \Rightarrow \bar{Q} = I - Q \in \mathbb{P}'.$$

But generally $\mathbb{P}'$ is not closed with respect to the operator product. Therefore we introduce the smaller set

$$\mathbb{P}'' = \{P \in \mathbb{P}' \colon PQ = QP\ (Q \in \mathbb{P}')\}. \tag{1.7}$$

$\mathbb{P}''$ possesses the following properties:

$$\begin{gathered} \mathbb{P} \subset \mathbb{P}'' \subset \mathbb{P}'; \\ P \in \mathbb{P}'' \Rightarrow \bar{P} \in \mathbb{P}''; \\ P_1, P_2 \in \mathbb{P}'' \Rightarrow P_1 P_2 = P_2 P_1 \in \mathbb{P}''. \end{gathered} \tag{1.8}$$

$(\mathbb{P}'', \leqslant)$ is a complemented distributive lattice (i.e., a Boolean algebra) with respect to the order relation

$$P_1 \leqslant P_2 \Leftrightarrow P_2 P_1 = P_1 \qquad (P_1, P_2 \in \mathbb{P}''). \tag{1.9}$$

In particular we have

$$\inf\{P_1, P_2\} = P_1 P_2 \in \mathbb{P}'', \tag{1.10}$$

$$\sup\{P_1, P_2\} = P_1 + P_2 - P_1 P_2 = P_1 \oplus P_2 \in \mathbb{P}''. \tag{1.11}$$

$P_1 \oplus P_2$ is called the Boolean sum of $P_1, P_2$.

Next we consider special projections in $\mathbb{P}''$. By construction we have

$$T_{m_1 \cdots m_d} = P_1^{m_1} \cdots P_d^{m_d} \in \mathbb{P}''.$$

$T_{m_1 \cdots m_d}$ is the *projector of d-variate polynomial tensor product interpolation*; it possesses the simple representation

$$T_{m_1 \cdots m_d}(f) = \sum_{i_1=1}^{a_1(m_1)} \cdots \sum_{i_d=1}^{a_d(m_d)} f(x_{i_1,1}, \ldots, x_{i_d,d}) \prod_{u=1}^{d} f_{i_u,u}^{m_u}. \tag{1.12}$$

The interpolation properties are described by

$$T_{m_1 \cdots m_d}(f)(x_{j_1,1}, \ldots, x_{j_d,d}) = f(x_{j_1,1}, \ldots, x_{j_d,d})$$
$$(j_u = 1, \ldots, a_u(m_u);\ u = 1, \ldots, d). \tag{1.13}$$

The range $\operatorname{Im}(T_{m_1 \cdots m_d})$ of $T_{m_1 \cdots m_d}$ is a $d$-variate polynomial tensor product space of dimension $a_1(m_1) \cdots a_d(m_d)$:

$$\operatorname{Im}(T_{m_1 \cdots m_d}) = \operatorname{span}\{x_1^{k_1} \cdots x_d^{k_d} : k_u = 0, \ldots, a_u(m_u) - 1;\ u = 1, \ldots, d\}$$
$$=: \mathbb{P}^d_{a_1(m_1)-1, \ldots, a_d(m_d)-1}. \tag{1.14}$$

## 2. $d$-Variate Boolean Interpolation

In this section we shall use the concept of Boolean sum of commuting projectors to construct $d$-variate interpolation schemes for distributions of interpolation points having a more complex structure than those of the tensor product schemes.

We assume that

$$q \in \mathbb{N}, \qquad d \leqslant q.$$

Then the operator

$$B_{q,d} = \bigoplus_{m_1 + \cdots + m_d = q} T_{m_1 \cdots m_d}$$

is an element of the Boolean algebra $\mathbb{P}''$ of projectors constructed in Section 1. We will call $B_{q,d}$ the *projector of d-variate Boolean interpolation.* The interpolation properties of the interpolation scheme defined by $B_{q,d}$ are described in the following Proposition 1.

Proposition 1. *Suppose that $f \in C(\bar{D})$. Then*

$$B_{q,d}(f)(x_{j_1,1}, \ldots, x_{j_d,d}) = f(x_{j_1,1}, \ldots, x_{j_d,d})$$
$$(j_u = 1, \ldots, a_u(m_u);\ u = 1, \ldots, d;\ m_1 + \cdots + m_d = q); \tag{2.1}$$

*i.e., the set of interpolation points of $B_{q,d}$ is the union of the sets of interpolation points of $T_{m_1 \cdots m_d}$ with $m_1 + \cdots + m_d = q$.*

*Proof.* It follows from the lattice-theoretical construction of $B_{q,d}$ that

$$T_{m_1 \cdots m_d} \leqslant B_{q,d} \qquad (m_1 + \cdots + m_d = q),$$

i.e., we have

$$T_{m_1\cdots m_d} B_{q,d} = T_{m_1\cdots m_d} \qquad (m_1 + \cdots + m_d = q). \tag{2.2}$$

Using (1.13) and (2.2) we can conclude

$$\begin{aligned} f(x_{j_1,1},\ldots, x_{j_d,d}) &= T_{m_1\cdots m_d}(f)(x_{j_1,1},\ldots, x_{j_d,d}) \\ &= T_{m_1\cdots m_d} B_{q,d}(f)(x_{j_1,1},\ldots, x_{j_d,d}) \\ &= B_{q,d}(f)(x_{j_1,1},\ldots, x_{j_d,d}) \end{aligned}$$

which completes the proof of Proposition 1.

The range $\mathrm{Im}(B_{q,d})$ of $B_{q,d}$ (i.e., the invariance set of $B_{q,d}$ [1]) is described in Proposition 2.

PROPOSITION 2. *The invariance set of $B_{q,d}$ is given by*

$$\begin{aligned} V_{q,d} &:= \mathrm{Im}(B_{q,d}) \\ &= \mathrm{span}\{x_1^{k_1}\cdots x_d^{k_d}\colon k_u = 0,\ldots, a_u(m_u) - 1; u = 1,\ldots, d; m_1 + \cdots + m_d = q\} \\ &= \sum_{m_1+\cdots+m_d=q} \mathbb{P}^d_{a_1(m_1)-1,\ldots,a_d(m_d)-1}. \end{aligned} \tag{2.3}$$

*Proof.* Note first that for any two commuting projectors $P_1, P_2 \in \mathbb{P}''$ the relation

$$\mathrm{Im}(P_1 \oplus P_2) = \mathrm{Im}(P_1) + \mathrm{Im}(P_2)$$

is true. Taking into account (1.14) and the definition of $B_{q,d}$ we obtain

$$\begin{aligned} \mathrm{Im}(B_{q,d}) &= \sum_{m_1+\cdots+m_d=q} \mathrm{Im}(T_{m_1\cdots m_d}) \\ &= \sum_{m_1+\cdots+m_d=q} \mathrm{span}\{x_1^{j_1}\cdots x_d^{j_d}\colon j_u = 0,\ldots, a_u(m_u) - 1; u = 1,\ldots, d\} \\ &= \mathrm{span}\{x_1^{k_1}\cdots x_d^{k_d}\colon k_u = 0,\ldots, a_u(m_u) - 1; \\ &\qquad u = 1,\ldots, d; m_1 + \cdots + m_q = q\}, \end{aligned}$$

i.e., we have

$$V_{q,d} = \sum_{m_1+\cdots+m_d=q} \mathbb{P}^d_{a_1(m_1)-1,\ldots,a_d(m_d)-1}. \tag{2.4}$$

This completes the proof of Proposition 2. We consider a simple but instructive example. Suppose that

$$\begin{aligned} x_{i,u} &= i \qquad (i \in \mathbb{N};\ u = 1,\dots, d), \\ a_u(m) &= m \qquad (m \in \mathbb{N};\ u = 1,\dots, d), \\ D &= \mathbb{R}^d. \end{aligned} \tag{2.5}$$

Then the invariance set $V_{q,d}$ of $B_{q,d}$ is the linear space of "complete" polynomials of degree $\leqslant q-d$:

$$\begin{aligned} V_{q,d} &= \sum_{m_1+\cdots+m_d=q} \mathbb{P}^d_{m_1-1,\dots,m_d-1} \\ &= \operatorname{span}\{x_1^{k_1}\cdots x_d^{k_d}\colon k_u = 0,\dots, m_u-1;\ u=1,\dots,d;\ m_1+\cdots+m_d = q\} \\ &=: \mathbb{P}_{q-d,d}. \end{aligned} \tag{2.6}$$

The interpolant $B_{q,d}(f)$ of $f$ possesses the following interpolation properties:

$$\begin{gathered} B_{q,d}(f)(j_1,\dots, j_d) = f(j_1,\dots, j_d) \\ (j_u = 1,\dots, m_u;\ u = 1,\dots, d;\ m_1+\cdots+m_d = q). \end{gathered} \tag{2.7}$$

Thus, $B_{q,d}$ (with (2.5)) is the *projector of cardinal polynomial Lagrange interpolation on d-simplices.*

## 3. A Representation formula for $B_{q,d}$

The projector $T_{m_1\cdots m_d}$ of tensor product interpolation has a simple representation (1.12). It is the purpose of this section to derive an expression of $B_{q,d}$ in terms of the projectors $T_{m_1\cdots m_d}$. For this reason we shall prove some lemmas for arbitrary projectors in the Boolean algebra $\mathbb{P}''$.

First we recall that a sequence $(Q_j)_{j\in\mathbb{N}}$ is a *chain* iff

$$Q_j \leqslant Q_{j+1} \qquad (Q_j \in \mathbb{P}'';\ j \in \mathbb{N}).$$

LEMMA 1. *Suppose that $(Q_i^1)_{i\in\mathbb{N}}$, $(Q_i^2)_{i\in\mathbb{N}}$ are chains in $\mathbb{P}''$. Put*

$$Q_{r,2} = \bigoplus_{i+j=r} Q_i^1 Q_j^2 \qquad (r \in \mathbb{N}). \tag{3.1}$$

*Then $(Q_{r,2})_{r\in\mathbb{N}}$ is also a chain in $\mathbb{P}''$; moreover*

$$Q_{r,2} = \sum_{i+j=r} Q_i^1 Q_j^2 - \sum_{i+j=r-1} Q_i^1 Q_j^2. \tag{3.2}$$

(*Empty sums and empty Boolean sums are* 0 *by definition.*)

*Proof.* It follows from the lattice-theoretical construction of $Q_{r,2}$, $Q_{r+1,2}$ that

$$\begin{aligned} Q_{r+1,2} &= \sup\{Q_r^1 Q_1^2, Q_{r-1}^1 Q_2^2, ..., Q_2^1 Q_{r-1}^2, Q_1^1 Q_r^2\} \\ &\geqslant \sup\{Q_r^1 Q_1^2, Q_{r-1}^1 Q_2^2, ..., Q_2^1 Q_{r-1}^2\} \\ &\geqslant \sup\{Q_{r-1}^1 Q_1^2, Q_{r-2}^1 Q_2^2, ..., Q_1^1 Q_{r-1}^2\} \\ &= Q_{r,2} \end{aligned}$$

Next we have

$$\begin{aligned} Q_{r,2} &= (Q_{r-1}^1 Q_1^2 \oplus \cdots \oplus Q_2^1 Q_{r-2}^2) \oplus Q_1^1 Q_{r-1}^2 \\ &= Q_1^1 Q_{r-1}^2 - Q_1^1 Q_{r-1}^2 (Q_{r-1}^1 Q_1^2 \oplus \cdots \oplus Q_2^1 Q_{r-2}^2) \\ &\quad + Q_{r-1}^1 Q_1^2 \oplus \cdots \oplus Q_2^1 Q_{r-2}^2 \\ &= Q_1^1 Q_{r-1}^2 - Q_1^1 Q_{r-2}^2 + Q_{r-1}^1 Q_1^2 \oplus \cdots \oplus Q_3^1 Q_{r-3}^2 \oplus Q_2^1 Q_{r-2}^2 \\ &= Q_1^1 Q_{r-1}^2 - Q_1^1 Q_{r-2}^2 + Q_2^1 Q_{r-2}^2 - Q_2^1 Q_{r-3}^2 + Q_{r-1}^1 Q_1^2 \oplus \cdots \oplus Q_3^1 Q_{r-3}^2 \\ &= \cdots \\ &= \sum_{s=1}^{r-1} Q_s^1 Q_{r-s}^2 + \sum_{s=1}^{r-2} Q_s^1 Q_{r-1-s}^2. \end{aligned}$$

This completes the proof of Lemma 1.

For the following it is useful to introduce the "ordinary" sum operators

$$C_{r,d} = \sum_{i_1 + \cdots + i_d = r} Q_{i_1}^1 \cdots Q_{i_d}^d$$

$$(Q_i^u \in \mathbb{P}'', i \in \mathbb{N}; u = 1, ..., d). \tag{3.3}$$

Note that

$$C_{r,d} = 0 \qquad (r \leqslant d - 1).$$

Our next objective is to extend Lemma 1.

LEMMA 2. *Suppose that*

$$(Q_i^u)_{i \in \mathbb{N}} \qquad (u = 1, ..., d)$$

*are chains in* $\mathbb{P}''$. *Put*

$$Q_{r,d} = \bigoplus_{i_1 + \cdots + i_d = r} Q_{i_1}^1 \cdots Q_{i_d}^d \qquad (r \in \mathbb{N}). \tag{3.4}$$

*Then* $(Q_{r,d})_{r \in \mathbb{N}}$ *is also a chain in* $\mathbb{P}''$.

*Furthermore, the formula*

$$Q_{r,d}=\sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\sum_{i_1+\cdots+i_d=r-j}Q^1_{i_1}\cdots Q^d_{i_d}$$
$$=\sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}C_{r-j,d} \tag{3.5}$$

*is true.*

*Proof.* First we have

$$Q_{r,d}=\bigoplus_{j_{d-1}=d-1}^{r-1}\bigoplus_{j_{d}-2=d-2}^{j_{d-1}-1}\cdots\bigoplus_{j_2=2}^{j_3-1}\bigoplus_{j_1=1}^{j_2-1}Q^1_{j_1}Q^2_{j_2-j_1}\cdots Q^{d-1}_{j_{d-2}-j_{d-2}}Q^d_{r-j_{d-1}}$$
$$=\bigoplus_{j=d-1}^{r-1}\left(\bigoplus_{i_1+\cdots+i_{d-1}=j}Q^1_{i_1}\cdots Q^{d-1}_{i_{d-1}}\right)Q^d_{r-j},$$

i.e., we have

$$Q_{r,d}=\bigoplus_{j=1}^{r-1}Q_{j,d-1}Q^d_{r-j}=\bigoplus_{i+j=r}Q_{i,d-1}Q^d_j. \tag{3.6}$$

In view of Lemma 1 and (3.6) it follows by induction that $(Q_{r,d})_{r\in\mathbb{N}}$ is a chain in $\mathbb{P}''$.

Moreover, formula (3.2) yields ($Q_{j,d-1}=0$ ($j\leqslant d-2$))

$$Q_{r,d}=\sum_{j=d-1}^{r-1}Q_{j,d-1}Q^d_{r-j}-\sum_{j=d-1}^{r-2}Q_{j,d-1}Q^d_{r-1-j}. \tag{3.7}$$

Again we apply induction on $d$ and we obtain (see (3.3))

$$Q_{r,d}=\sum_{j=d-1}^{r-1}\left(\sum_{i=0}^{(d-1)-1}(-1)^i\binom{(d-1)-1}{i}C_{j-i,d-1}\right)Q^d_{r-j}$$
$$-\sum_{j=d-1}^{r-2}\left(\sum_{i=0}^{(d-1)-1}(-1)^i\binom{(d-1)-1}{i}C_{j-i,d-1}\right)Q^d_{r-1-j}$$
$$=\sum_{i=0}^{(d-1)-1}(-1)^i\binom{(d-1)-1}{i}\sum_{j=d-1}^{r-1}C_{j-i,d-1}Q^d_{r-j}$$
$$-\sum_{i=0}^{(d-1)-1}(-1)^i\binom{(d-1)-1}{i}\sum_{j=d-1}^{r-2}C_{j-i,d-1}Q^d_{r-1-j}$$
$$=\sum_{i=0}^{(d-1)-1}(-1)^i\binom{(d-1)-1}{i}C_{r-i,d}$$
$$-\sum_{i=0}^{(d-1)-1}(-1)^i\binom{(d-1)-1}{i}C_{r-1-i,d}$$

$$
\begin{aligned}
&= \sum_{i=0}^{(d-1)-1} (-1)^i \binom{(d-1)-1}{i} C_{r-i,d} \\
&\quad - \sum_{k=1}^{d-1} (-1)^{k-1} \binom{(d-1)-1}{k-1} C_{r-k,d} \\
&= C_{r,d} \\
&\quad + \sum_{k=1}^{(d-1)-1} (-1)^k \left( \binom{(d-1)-1}{k} + \binom{(d-1)-1}{k-1} \right) C_{r-k,d} \\
&\quad + (-1)^{d-1} C_{r-(d-1),d} \\
&= \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} C_{r-k,d}
\end{aligned}
$$

which completes the proof of Lemma 2.

THEOREM 1. *The projector of d-variate Boolean interpolation, $B_{q,d}$, possesses the following representation*:

$$
\begin{aligned}
B_{q,d} &= \bigoplus_{m_1+\cdots+m_d=q} T_{m_1\cdots m_d} \\
&= \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \sum_{n_1+\cdots+n_d=q-j} T_{n_1\cdots n_d}.
\end{aligned} \tag{3.8}
$$

*Proof.* Since $(P_u^i)_{i\in\mathbb{N}}$ $(u=1,\dots,d)$ are chains in $\mathbb{P}''$ and $T_{n_1\cdots n_d} = P_1^{n_1}\cdots P_d^{n_d}$ an application of Lemma 2 yields (3.8).

For $d=2$ we have

$$
B_{q,2} = \sum_{i+j=q} T_{ij} - \sum_{i+j=q-1} T_{ij}; \tag{3.9}
$$

for $d=3$ we obtain

$$
\begin{aligned}
B_{q,3} = &\sum_{i+j+k=q} T_{ijk} \\
&-2 \sum_{i+j+k=q-1} T_{ijk} \\
&+ \sum_{i+j+k=q-2} T_{ijk}.
\end{aligned} \tag{3.10}
$$

## 4. Fundamental Functions of $d$-Variate Boolean Interpolation

In this section we shall apply the representation formula of $B_{q,d}$ to give an explicit expression for the *fundamental functions* (or cardinal functions) *of d-variate Boolean interpolation.* We define

$$F^q_{j_1 \cdots j_d} = B_{q,d}(f^{q-d+1}_{j_1,1} \cdots f^{q-d+1}_{j_d,d})$$
$$(j_u = 1,\ldots, a_u(m_u);\ u = 1,\ldots, d;\ m_1 + \cdots + m_d = q). \tag{4.1}$$

It follows from Proposition 1 and the properties of $f^{q-d+1}_{j_u,u}$ $(u = 1,\ldots, d)$ that the functions $F^q_{j_1 \cdots j_d}$ satisfy the *cardinality relations*

$$F^q_{j_1 \cdots j_d}(x_{i_1,1},\ldots, x_{i_d,d}) = \delta_{i_1,j_1} \cdots \delta_{i_d,j_d}$$
$$(i_u = 1,\ldots, a_u(m_u);\ u = 1,\ldots, d;\ m_1 + \cdots + m_d = q;$$
$$j_v = 1,\ldots, a_v(n_v);\ \ v = 1,\ldots, d;\ \ n_1 + \cdots + n_d = q). \tag{4.2}$$

Furthermore, Proposition 2 yields

$$F^q_{j_1 \cdots j_d} \in \operatorname{span}\{x_1^{k_1} \cdots x_d^{k_d}:$$
$$k_u = 0,\ldots, a_u(m_u) - 1;\ u = 1,\ldots, d;\ m_1 + \cdots + m_d = q\}$$
$$(j_v = 1,\ldots, a_v(n_v);\ v = 1,\ldots, d;\ n_1 + \cdots + n_d = q). \tag{4.3}$$

It follows from (4.2), (4.3) that

$$\{F^q_{j_1 \cdots j_d}: j_v = 1,\ldots, a_v(n_v);\ v = 1,\ldots, d;\ n_1 + \cdots + n_d = q\}$$

is a basis of $\operatorname{Im}(B_{q,d})$, the *dual basis* of the functionals

$$L_{i_1 \cdots i_d}(f) = f(x_{i_1,1},\ldots, x_{i_d,d})$$
$$(i_u = 1,\ldots, a_u(m_u);\ u = 1,\ldots, d;\ m_1 + \cdots + m_d = q) \tag{4.4}$$

with respect to the sbuspace $\operatorname{Im}(B_{q,d})$. Thus we can expand the Boolean interpolant $B_{q,d}(f)$ of $f$ in terms of the functions $F^q_{j_1 \cdots j_d}$:

Proposition 3. *Put* $a_u(0) := 0$ $(u = 1,\ldots, d)$. *Then we have for any* $f \in C(\bar{D})$

$$B_{q,d}(f) = \sum_{r=d}^{q} \sum_{s_{d-1}=d-1}^{r-1} \cdots \sum_{s_2=2}^{s_3-1} \sum_{s_1=1}^{s_2-1}$$

$$\sum_{k_1=a_1(s_1-1)+1}^{a_1(s_1)} \sum_{k_2=a_2(s_2-s_1-1)+1}^{a_2(s_2-s_1)} \cdots \sum_{k_d=a_d(r-s_{d-1}-1)+1}^{a_d(r-s_{d-1})} f(x_{k_1,1},\ldots, x_{k_d,d})\, F^q_{k_1\cdots k_d}. \tag{4.5}$$

Our next objective is to derive an explicit expression for the fundamental functions $F^q_{k_1\cdots k_d}$ in terms of the *fundamental functions of tensorproduct interpolation*:

$$f^{n_1\cdots n_d}_{k_1\cdots k_d} = f^{n_1}_{k_1,1}\cdots f^{n_d}_{k_1,d}$$

$$(k_u = 1,\ldots, a_u(n_u);\ u = 1,\ldots, d). \tag{4.6}$$

THEOREM 2. *Assume that*

$$a_u(m_u-1) < k_u \leqslant a_u(m_u) \qquad (u = 1,\ldots, d). \tag{4.7}$$

*Then we have*

$$F^q_{k_1\cdots k_d} = \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \sum_{\substack{n_1+\cdots+n_d=q-j\\ m_1\leqslant n_1,\ldots,m_d\leqslant n_d}} f^{n_1 n_d}_{k_1\cdots k_d} \tag{4.8}$$

$$= \sum_{j=0}^{d-1} (-)^j \binom{d-1}{j} \sum_{s_{d-1}=m_{d-1}+\cdots+m_1}^{q-j-m_d} \sum_{s_{d-2}=m_{d-2}+\cdots+m_1}^{s_{d-1}-m_{d-1}} \cdots \sum_{s_2=m_2+m_1}^{s_3-m_3} \sum_{s_1=m_1}^{s_2-m_2} f^{s_1}_{k_1,1} f^{s_2-s_1}_{k_2,2}\cdots f^{s_{d-1}-s_{d-2}}_{k_{d-1},d-1} f^{q-j-s_{d-1}}_{k_d,d}. \tag{4.9}$$

*Proof.* Using Theorem 1 and (1.12) we obtain

$$
\begin{aligned}
F^q_{k_1\cdots k_d} &= B_{q,d}(f^{q-d+1}_{k_1,1}\cdots f^{q-d+1}_{k_d,d})\\
&= \sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\sum_{n_1+\cdots+n_d=q-j} T_{n_1\cdots n_d}\left(\prod_{u=1}^{d} f^{q-d+1}_{k_u,u}\right)\\
&= \sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\sum_{\substack{n_1+\cdots+n_d=q-j\\ n_1\geqslant m_1,\ldots,n_d\geqslant m_d}} f^{n_1}_{k_1,1}\cdots f^{n_d}_{k_d,d}
\end{aligned}
$$

whence (4.8) follows.

We define $s_1,\ldots,s_{d-1}$ by

$$
\begin{aligned}
n_1 &= s_1,\\
n_2 &= s_2 - s_1,\\
&\vdots\\
n_{d-1} &= s_{d-1} - s_{d-2},\\
n_d &= q - j - s_{d-1}.
\end{aligned}
$$

From (4.7) it follows that

$$
\begin{aligned}
m_1 &\leqslant s_1,\\
m_2 &\leqslant s_2 - s_1,\\
&\vdots\\
m_{d-1} &\leqslant s_{d-1} - s_{d-2},\\
m_d &\leqslant q - j - s_{d-1}.
\end{aligned}
$$

Then we can conclude

$$
\begin{aligned}
F^q_{k_1\cdots k_d} &= \sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\sum_{\substack{n_1+\cdots+n_d=q-j\\ n_1\geqslant m_1,\ldots,n_d\geqslant m_d}} f^{n_1}_{k_1,1}\cdots f^{n_d}_{k_d,d}\\
&= \sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\\
&\qquad \sum_{\substack{s_{d-1}=d-1\\ m_d\leqslant q-j-s_{d-1}}}^{q-j-1}\\
&\qquad \sum_{\substack{s_{d-2}=d-2\\ m_{d-1}\leqslant s_{d-1}-s_{d-2}}}^{s_{d-1}-1}\\
&\qquad \vdots
\end{aligned}
$$

$$\sum_{\substack{s_2=2\\ m_2\leqslant s_2-s_1}}^{s_3-1}\ \sum_{\substack{s_1=1\\ m_1\leqslant s_1}}^{s_2-1} f_{k_1,1}^{s_1} f_{k_2,2}^{s_2-s_1}\cdots f_{k_{d-1},d-1}^{s_{d-1}-s_{d-2}} f_{k_d,d}^{q-j-s_{d-1}}$$

$$= \sum_{j=0}^{d-1} (-1)^j \binom{d-1}{j} \sum_{s_{d-1}=m_{d-1}+\cdots+m_1}^{q-j-m_d}\ \sum_{s_{d-2}=m_{d-2}+\cdots+m_1}^{s_{d-1}-m_{d-1}} \cdots \sum_{s_2=m_2+m_1}^{s_3-m_3}\ \sum_{s_1=m_1}^{s_2-m_2} f_{k_1,1}^{s_1} f_{k_2,2}^{s_2-s_1}\cdots f_{k_{d-1},d-1}^{s_{d-1}-s_{d-2}} f_{k_d,d}^{q-j-s_{d-1}}$$

which completes the proof of Theorem 2.

We conclude this section by considering *cardinal polynomial Lagrange interpolation on d-simplices* (see (2.5)–(2.7)). In view of (4.5) we have

$$B_{q,d}(f) = \sum_{r=d}^{q}\ \sum_{s_{d-1}=d-1}^{r-1} \cdots \sum_{s_2=2}^{s_3-1}\ \sum_{s_1=1}^{s_2-1} f(s_1, s_2-s_1,\dots, r-s_{d-1})\, F^q_{s_1,s_2-s_1,\dots,r-s_{d-1}}. \tag{4.10}$$

We define

$$\begin{aligned} m_1 &= s_1,\\ m_2 &= s_2 - s_1,\\ &\vdots\\ m_{d-1} &= s_{d-1}-s_{d-2},\\ m_d &= r - s_{d-1} \end{aligned}$$

and we obtain a more geometric formula for $B_{q,d}(f)$:

$$B_{q,d}(f)=\sum_{\substack{d\leqslant m_1+\cdots+m_d\leqslant q\\ 1\leqslant m_1,\ldots,1\leqslant m_d}} f(m_1,\ldots,m_d)\,F^q_{m_1,\ldots,m_d}. \tag{4.11}$$

The fundamental functions are obtained from (4.8), (4.9):

$$\begin{aligned} F^q_{m_1\cdots m_d} &= \sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\sum_{\substack{n_1+\cdots+n_d=q-j\\ m_1\leqslant n_1,\ldots,m_d\leqslant n_d}} f^{n_1 n_d}_{m_1\cdots m_d}\\ &= \sum_{j=0}^{d-1}(-1)^j\binom{d-1}{j}\sum_{s_{d-1}=m_{d-1}+\cdots+m_1}^{q-j-m_d}\\ &\;\vdots\\ &\sum_{s_2=m_2+m_1}^{s_3-m_3}\ \sum_{s_1=m_1}^{s_2-m_2} f^{s_1}_{m_1,1} f^{s_2-s_1}_{m_2,2}\cdots f^{q-j-s_{d-1}}_{m_d,d}. \end{aligned} \tag{4.12}$$

## 5. A Trivariate Example

To illustrate the method of Boolean interpolation we construct a *trivariate "serendipity" family* of even degree (see also [3, 9]). The points $x_{i,u}$ ($i\in\mathbb{N}$, $u=1,2,3$) are chosen as follows:

$$\begin{aligned} x_{1,u}&=-1,\\ x_{2,u}&=+1,\\ x_{3,u}&=0,\\ x_{4,u}&=1-2^{-1},\\ x_{5,u}&=-1+2^{-1},\\ x_{6,u}&=1-2^{-2},\\ x_{7,u}&=-1+2^{-2},\ldots, \end{aligned}$$

i.e., we have

$$\begin{gathered} x_{1,u}=-1,\qquad x_{2,u}=+1,\qquad x_{3,u}=0,\\ x_{2k,u}=1-2^{-(k-1)},\qquad x_{2k+1,u}=-1+2^{-(k-1)}\\ (k\in\mathbb{N},\ k\geqslant 2),\qquad u=1,2,3. \end{gathered} \tag{5.1}$$

The functions $a_u$ are defined as follows:

$$a_u(1) = 2,$$
$$a_u(2) = 3,$$
$$a_u(m) = 2m - 1 \qquad (m \in \mathbb{N},\ m \geqslant 3)$$
$$(u = 1, 2, 3). \tag{5.2}$$

We consider the interpolation scheme induced by $B_{5,3}$ in greater detail. The nodal configuration of $B_{5,3}$ is presented in Fig. 1. We use Theorem 2 to compute the fundamental functions for the typical points $Z_1$, $Z_2$, $Z_3$, $Z_4$ of the trivariate Scott-element. Note that this element has no interior points.

$$Z_1 = (x_{1,1}, x_{1,2}, x_{1,3})$$

$$\begin{aligned} F^5_{1,1,1} = {} & f^1_{1,1} f^1_{1,2} f^3_{1,3} \\ & + (f^1_{1,1} f^2_{1,2} + f^2_{1,1} f^1_{1,2}) f^2_{1,3} \\ & + (f^1_{1,1} f^3_{1,2} + f^2_{1,1} f^2_{1,2} + f^3_{1,1} f^1_{1,2}) f^1_{1,3} \\ & - 2(f^1_{1,1} f^1_{1,2} f^2_{1,3} + (f^1_{1,1} f^2_{1,2} + f^2_{1,1} f^1_{1,2}) f^1_{1,3}) \\ & + f^1_{1,1} f^1_{1,2} f^1_{1,3}. \end{aligned}$$

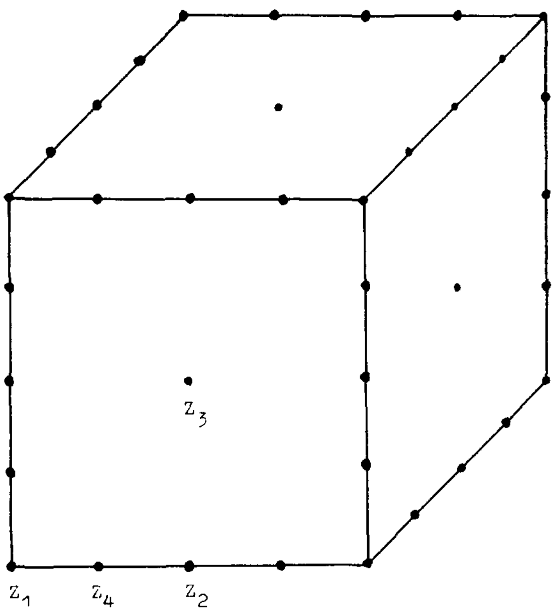


FIG. 1. Trivariate Scott-element.

$$Z_2 = (x_{3,1}, x_{1,2}, x_{1,3})$$

$$F^5_{3,1,1} = f^2_{3,1} f^1_{1,2} f^2_{1,3} + (f^2_{3,1} f^2_{1,2} + f^3_{3,1} f^1_{1,2}) f^1_{1,3} - 2 f^2_{3,1} f^1_{1,2} f^1_{1,3}.$$

$$Z_3 = (x_{3,1}, x_{1,2}, x_{3,3})$$

$$F^5_{3,1,3} = f^2_{3,1} f^1_{1,2} f^2_{3,3}.$$

$$Z_4 = (x_{5,1}, x_{1,2}, x_{1,3})$$

$$F^5_{5,1,1} = f^3_{5,1} f^1_{1,2} f^1_{1,3}.$$

Finally we note that the invariance set of $B_{5,3}$ is given by

$$\mathrm{Im}(B_{5,3}) = \mathbb{P}^3_{1,1,4} + \mathbb{P}^3_{1,2,2} + \mathbb{P}^3_{1,4,1} + \mathbb{P}^3_{2,1,2} + \mathbb{P}^3_{2,2,1} + \mathbb{P}^3_{4,1,1}.$$

Therefore, $\mathrm{Im}(B_{5,3})$ contains the space of complete trivariate polynomials of total degree $\leqslant 4$:

$$\mathrm{Im}(B_{5,3}) \supset \mathbb{P}_{4,3}.$$

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